Problem 26. Show that the polar decomposition is unique: If $T \in B(H)$ can be written as $T = UP$ with $P \geq 0$ and a partial isometry U such that $\ker U = \ker P$, then $P = (T^*T)^{1/2}$ and U are uniquely determined.

Problem 27. Let $T \in B(H)$. Show that T is invertible if and only if T^*T and TT^* are bounded from below.

Problem 28. Let $P, Q \in B(H)$ be orthogonal projections and denote $U = \operatorname{ran} P$, $V = \operatorname{ran} Q$. Recall that $P \wedge Q :=$ projection onto $U \cap V$ and $P \vee Q :=$ projection onto $\overline{U + V}$. Show the following:

(i) $PQ = QP$ implies $P \wedge Q = PQ$ and $P \vee Q = P + Q - PQ$,
(ii) $P \leq Q$ if and only if $PQ = QP = P$ if and only if $\operatorname{ran} P \subseteq \operatorname{ran} Q$,
(iii) $PQ = 0$ if and only if $P \vee Q = P + Q$.

Problem 29. Let E be a spectral measure on $(\Omega, \mathfrak{M})$, $B(\Omega, \mathfrak{M})$ be the Banach space of bounded $\mathfrak{M}$-measurable functions on Ω and B_s denote the subspace of simple functions in $B(\Omega, \mathfrak{M})$. Hence, any $f \in B_s$ can be written in the form

$$f = \sum_{r=1}^{n} c_r \chi_{M_r},$$

where $c_1, \ldots, c_n \in \mathbb{C}$ and $M_1, \ldots, M_n$ are pairwise disjoint sets in $\mathfrak{M}$. For such a function we write

$$\mathbb{I}(f) = \sum_{r=1}^{n} c_r E(M_r).$$

For $f, g \in B(\Omega, \mathfrak{M})$, $\alpha, \beta \in \mathbb{C}$ and $x, y \in H$, show the following:

(i) $\mathbb{I}(f)$ is well-defined,
(ii) $\mathbb{I}(\overline{f}) = \mathbb{I}(f)^*$
$\mathbb{I}(\alpha f + \beta g) = \alpha \mathbb{I}(f) + \beta \mathbb{I}(g)$
$\mathbb{I}(fg) = \mathbb{I}(f)\mathbb{I}(g)$,
(iii) $\langle \mathbb{I}(f)x, y \rangle = \int_\Omega f(t) d\langle E(t)x, y \rangle$,
(iv) $||\mathbb{I}(f)x||^2 = \int_\Omega |f(t)|^2 d\langle E(t)x, x \rangle$,
(v) Let $f_n \in B(\Omega, \mathfrak{M})$ for $n \in \mathbb{N}$. If $f_n(t) \to f(t)$ E-a.e. on Ω and $\sup_n ||f_n||_\infty < \infty$, then $\text{SOT-}\lim_{n\to\infty} \mathbb{I}(f_n) = \mathbb{I}(f)$.